

# Directed Percolation with a Conserved Field and the Depinning Transition

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Conserved directed-percolation (C-DP) and the depinning transition of a disordered elastic interface belong to the same universality class as has been proven very recently by Le Doussal and Wiese [Phys. Rev. Lett. **114**, 110601 (2015)] through a mapping of the field theory for C-DP onto that of the quenched Edwards-Wilkinson model. Here, we present an alternative derivation of the C-DP field theoretic functional, starting with the coherent state path integral formulation of the C-DP and then applying the Grassberger-transformation, that avoids the disadvantages of the so-called Doi-shift. We revisit the aforementioned mapping with focus on a specific term in the field theoretic functional that has been problematic in the past when it came to assessing its relevance. We show that this term is redundant in the sense of the renormalization group.

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## I. INTRODUCTION

About 20 years ago, van Wijland, Oerding, and Hilhorst [1] introduced a model of the propagation of an epidemic in a population of fluctuating density. Healthy (inactive) and sick (active) individuals, also called  $A$  and  $B$  particles, diffuse freely and independently on a lattice of dimension  $d$ , and react as  $A + B \rightarrow 2B$ ,  $B \rightarrow A$ , therefore holding the total number of particles globally constant. As long as the diffusion constants of both particle types are non-zero, a consistently renormalizable field theory, commonly referred to as *directed percolation with a conserved field* (DP-C), can be derived, and the universal scaling properties can be calculated in an  $\varepsilon$ -expansion [2–4]. The model features a continuous, a tricritical [5], and a fluctuation induced discontinuous transition [2] depending on the ratio of the diffusion constants. A variation of the model where only the active individuals (the agent) can diffuse, has become to known as the *conserved directed-percolation* (C-DP) model. In contrast to DP-C, C-DP has proven to be notoriously difficult when it comes to renormalized field theory [6]. Numerical studies, however, have been fruitful, and it has been established that C-DP and certain sandpile-models belong to the same universality class, the so called Manna-class of self-organized criticality, and it was argued that, *inter alia*, the depinning model of interfaces in random media belongs to this class too [7–14]. Very recently, Le Doussal and Wiese (LeDW) [15] proved the latter by presenting an exact mapping of C-DP to the quenched Edwards-Wilkinson (qEW) model that describes the depinning transition of a disordered elastic interface [16–19].

In this paper, we revisit the work of LeDW and we shed light on the mapping from C-DP to qEW from somewhat different angles. First, we re-derive the field theoretic response functional for C-DP (the starting point of the mapping) using the so-called Grassberger transformation, an approach that, as we think, is more natural

than that taken in previous derivations in that the actual number densities of particles serve as field variables and that avoids problematic physical interpretations of imaginary fields. Then we discuss the mapping in a way that is slightly different from that of LeDW with focus on a specific term in the field theoretic functional. LeDW showed that this term primarily changes the friction coefficient and they claimed that the ensuing remaining term is *irrelevant*. We show that this term is *redundant* in the sense of the renormalization group (RG) [20, 21], i.e., it does not need its own renormalization and hence it does not impact the scaling behavior.

## II. MODEL AND FIELD THEORETIC FUNCTIONAL

The C-DP model is based on the reactions

$$A + B \leftrightarrow 2B, \quad (2.1a)$$

$$B \rightarrow A. \quad (2.1b)$$

Note that Eq. (2.1a) includes both, forth and back reactions. The  $B$ -particles diffuse in a  $d$ -dimensional volume whereas the  $A$ -particles are immobile. It is well known, that the corresponding coherent state path integral (CSPI)-action [22–27] is given by

$$\begin{aligned} \mathcal{S} = \int d^d x \int_{-\infty}^{\infty} dt \big\{ & (\hat{a} - 1) \partial_t a + (\hat{b} - 1) \partial_t b + D \nabla \hat{b} \cdot \nabla b \\ & - k_1 (\hat{b}^2 - \hat{a} \hat{b}) ab - k_2 (\hat{a} \hat{b} - \hat{b}^2) b^2 - k_3 (\hat{a} - \hat{b}) b \big\}, \end{aligned} \quad (2.2)$$

where  $a, \hat{a}$  and  $b, \hat{b}$  are the coherent fields describing the species  $A$  and  $B$ , respectively, that are subject to the initial and final conditions  $\hat{a}(\infty) = \hat{b}(\infty) = 1$ ,  $a(-\infty) = b(-\infty) = 0$ . It is important to note that these fields

are in general complex and do not correspond to particle densities. These densities are given by  $n_A = \hat{a}a$  and  $n_B = \hat{b}b$  which are real and non-negative by construction. To proceed from the CSPI-action (2.2) to a field theoretic functional, previous studies [26, 27] have relied on the so-called Doi-shift  $\hat{a} = 1 + \tilde{a}$ ,  $\hat{b} = 1 + \tilde{b}$ . This approach has the disadvantage that it produces, in a Langevin interpretation, mixed real and imaginary noise, the latter resulting from the annihilation of  $A$ -particles in reaction (2.1a). Furthermore, this approach has the disadvantage that masks the conservation property of the total number of particles  $A$  and  $B$  since the coherent fields  $a$  and  $b$  are not particle densities. As we will show, these problems can be avoided by switching to a description based on the particle densities with help of the quasicanonical Grassberger-transformation (see the Appendix for details and background information)

$$\hat{a} = \exp(\tilde{n}_A), \quad a = n_A \exp(-\tilde{n}_A), \quad (2.3a)$$

$$\hat{b} = \exp(\tilde{n}_B), \quad b = n_B \exp(-\tilde{n}_B), \quad (2.3b)$$

with  $\tilde{n}_A(\infty) = \tilde{n}_B(\infty) = 0$  and  $n_A(-\infty) = n_B(-\infty) = 0$ . When source fields  $\rho_A$  and  $\rho_B$  that feed additional particles into the system are admitted, the resulting action is

$$\begin{aligned} \mathcal{A} = \int d^d x \int_{-\infty}^{\infty} dt \Big\{ & \tilde{n}_A \partial_t n_A + \tilde{n}_B \partial_t n_B \\ & + D(\nabla \tilde{n}_B \cdot \nabla n_B - n_B(\nabla \tilde{n}_B)^2) - \rho_A \tilde{n}_A - \rho_B \tilde{n}_B \\ & - (\exp(\tilde{n}_B - \tilde{n}_A) - 1) k_1 n_A n_B \\ & - (\exp(\tilde{n}_A - \tilde{n}_B) - 1) (k_2 n_B^2 + k_3 n_B) \Big\}. \end{aligned} \quad (2.4)$$

This action guarantees the conservation of the total particle density  $n_A(\mathbf{x}, t) + n_B(\mathbf{x}, t)$ , which can be seen as follows. We demand the invariance of  $\mathcal{A}$  under the symmetry transformation

$$\tilde{n}_A(\mathbf{x}, t) \mapsto \tilde{n}_A(\mathbf{x}, t) + \phi(t), \quad (2.5a)$$

$$\tilde{n}_B(\mathbf{x}, t) \mapsto \tilde{n}_B(\mathbf{x}, t) + \phi(t) \quad (2.5b)$$

for any purely time-dependent function  $\phi(t)$  with  $\phi(\infty) = 0$ . This symmetry transformation implies that

$$\begin{aligned} \frac{d}{dt} \int d^d x (n_A(\mathbf{x}, t) + n_B(\mathbf{x}, t)) \\ = \int d^d x (\rho_A(\mathbf{x}, t) + \rho_B(\mathbf{x}, t)), \end{aligned} \quad (2.6)$$

i.e., the average particle density is constant if particle sources are absent. Note that the conservation-symmetry transformation Eqs. (2.5) simply changes the density response fields with an additive contribution that is linear in the generator  $\phi$ . In comparison, the corresponding transformation in the formulation based on the Doi-shift is very clumsy. Hence, also in this respect, the density variables are advantageous over the coherent fields: they make transparent the role of the response field as the generators of the particle-conservation symmetry.

Now we turn to the scaling behavior of the fields under coarse graining. The fields  $\tilde{n}_A$ ,  $n_A$ ,  $\tilde{n}_B$  and  $n_B$  can be rescaled so that all of them attain a positive scaling dimension (see Appendix A and the argumentation below). Hence, we can truncate the expansion of the exponentials in Eq. (2.4) after the quadratic term, and we can neglect the irrelevant diffusional noise as well other irrelevant contributions. After letting  $n_A + n_B \rightarrow c$ ,  $\tilde{n}_A \rightarrow \tilde{c}$ ,  $n_B \rightarrow n$ ,  $\tilde{n}_B - \tilde{n}_A \rightarrow \tilde{n}$ ,  $\rho_A + \rho_B \rightarrow \rho$ , and  $\rho_B \rightarrow \sigma$  so that  $n$  is now the particle density of the agent and  $\sigma$  is the corresponding source field, we obtain the dynamical response functional [31–33]

$$\begin{aligned} \mathcal{J} = \int d^d x \int_{-\infty}^{\infty} dt \lambda \Big\{ & \tilde{n} \lambda^{-1} \partial_t n - \tilde{n} \nabla^2 n + \tilde{c} \lambda^{-1} \partial_t c \\ & - \tilde{c} \nabla^2 n - \rho \tilde{c} - \sigma \tilde{n} - \tilde{n} (gc - fn - \kappa) n - \frac{1}{2} \tilde{n}^2 n \Big\}. \end{aligned} \quad (2.7)$$

For an alternate derivation of this functional based on the Doi-shift, see Appendix B.

Note that  $\mathcal{J}$  corresponds to the well known coarse grained effective Langevin description (in Ito-interpretation and without the sources)

$$\lambda^{-1} \partial_t n = \nabla^2 n + (gc - fn - \kappa) n + \eta, \quad (2.8a)$$

$$\lambda^{-1} \partial_t c = \nabla^2 n, \quad (2.8b)$$

$$\langle \eta(\mathbf{x}, t) \rangle = 0, \quad (2.8c)$$

$$\langle \eta(\mathbf{x}, t) \eta(\mathbf{x}', t') \rangle = \lambda n(\mathbf{x}, t) \delta(\mathbf{x} - \mathbf{x}') \delta(t - t'), \quad (2.8d)$$

of C-DP. Note also that  $f = g$  if the back reaction in (2.1a) is forbidden.

Dimensional analysis shows that the various quantities superficially scale in terms of an appropriate inverse length-scale  $\mu$  as  $(\tilde{n}, \tilde{c}, \kappa) \sim \mu^2$ ,  $(n, c) \sim \mu^{d-2}$ ,  $(f, g) \sim \mu^{4-d}$  if  $\mathbf{x} \sim \mu^{-1}$  and  $\lambda t \sim \mu^{-2}$ . This signals that  $d_c = 4$  is the upper critical dimension of the absorbing transition below which both coupling constants  $f$  and  $g$  become relevant. It can be easily shown that the response functional  $\mathcal{J}$  encompasses all relevant operators. All other operators, i.e., all other monomials that can be constructed from the fields  $n, \tilde{n}$  and  $c, \tilde{c}$  and their derivatives, are irrelevant since their superficial scaling dimensions are larger than  $d + 2$ , and the corresponding coupling constants have negative scaling dimensions lower than  $-2$  near  $d = d_c = 4$ . In particular the diffusional noise  $n(\nabla \tilde{n})^2$  is irrelevant.

Alternatively, using  $m = c - n$  instead of  $c$ , Eqs. (2.8) can be recast as

$$\lambda^{-1} \partial_t n = \nabla^2 n - \lambda^{-1} \partial_t m, \quad (2.9a)$$

$$\lambda^{-1} \partial_t m = -(gm - (f - g)n - \kappa)n - \eta. \quad (2.9b)$$

This form suggests that the time-integrated local agent density  $\int dt n \sim s$  is another potentially useful choice for the independent density-field instead of  $n$  itself. Indeed, this field is an essential ingredient in the LeDW mapping of C-DP to qEW, see below.

In the following, we consider processes beginning at some time  $t > 0$ . Hence,  $\sigma(\mathbf{x}, t)$  and  $n(\mathbf{x}, t)$  are zero if  $t \leq 0$ . We assume that the inactive particles are placed into the system homogeneously with density  $c_0$  at some time  $t_0 < 0$ . Therefore  $\rho(\mathbf{x}, t) = c_0\delta(t - t_0) + \sigma(\mathbf{x}, t)$ , and  $c(\mathbf{x}, t) = c_0\theta(t - t_0)$  for  $t < 0$ . Hence the time integral in Eq. (2.7) can be reduced to only positive times with an initial condition  $c(\mathbf{x}, 0) = c_0$ .

To establish closer contact to the work by LeDW, we now let  $\tilde{c} + \tilde{n} \rightarrow \tilde{n}'$ ,  $\tilde{n} \rightarrow g\tilde{\zeta}$ ,  $m = c - n \rightarrow g^{-1}(\kappa - \zeta)$ , and obtain

$$\mathcal{J} = \int d^d x \int_0^\infty dt \lambda \left\{ \tilde{n}' (\lambda^{-1} \partial_t n - \nabla^2 n - g^{-1} \lambda^{-1} \partial_t \zeta) + gn [ \tilde{\zeta} ((\lambda gn)^{-1} \partial_t \zeta + \zeta + (f - g)n) - \frac{g}{2} \tilde{\zeta}^2 ] \right\}, \quad (2.10)$$

where  $\zeta(\mathbf{x}, 0) = \kappa - gc_0$ . This is the starting point of the LeDW mapping from C-DP to qEW. In this representation of the response functional, diffusional motion is formally separated from the local fluctuations described by the fields  $\zeta$  and  $\tilde{\zeta}$ .

### III. THE LE-DOUSSAL-WIESE MAPPING OF C-DP TO QEW

The essential tool of the LeDW mapping is the switch from the local agent-density  $n$  to its the time-integrated version  $s$  which can be viewed as an interface-height. Taking the same route, we define the new field and its conjugated response field by

$$s(\mathbf{x}, t) = \lambda g \int_0^t dt' n(\mathbf{x}, t'), \quad (3.1a)$$

$$\tilde{s}(\mathbf{x}, t) = -\frac{1}{\lambda g} \frac{\partial \tilde{n}'(\mathbf{x}, t)}{\partial t}. \quad (3.1b)$$

Because we are approaching the C-DP transition from the active side where  $n(t) > 0$ , it is guaranteed that  $s$  is monotonically increasing in  $t$ , and it can therefore be used as a local time variable  $t \rightarrow t(s, \mathbf{x})$ , with increment  $ds = \lambda gn dt$ , as long as  $\mathbf{x}$  is held constant. The introduction of the new field transforms the response functional (2.10) to

$$\mathcal{J} = \int d^d x \left\{ \int_0^\infty dt \lambda \tilde{s} (\lambda^{-1} \partial_t s - \nabla^2 s - k - \zeta) + \int_0^\infty ds \left[ \tilde{\zeta} (\partial_s \zeta + \zeta) - \frac{g}{2} \tilde{\zeta}^2 + \tilde{\zeta} \alpha \lambda^{-1} \dot{s} \right] \right\}, \quad (3.2)$$

where  $\alpha = f/g - 1 \geq 0$ ,  $k = gc_0 - \kappa$ , and  $c_0$  now denotes the density of all, active and inactive, particles initially. As functions of  $s$  instead of  $t$ , the fields  $\tilde{\zeta}$  and  $\zeta$  appear in the path integral with weight  $\exp(-\mathcal{J})$  in Gaussian form, and describe an Ornstein-Uhlenbeck-process. They

easily are integrated out leading to the reduced response functional [34]

$$\mathcal{J}_{red} = \int d^d x \left\{ \int_0^\infty dt \lambda \tilde{s}(t) [\lambda^{-1} \dot{s}(t) - \nabla^2 s(t) - k] + \int_0^\infty dt dt' [\alpha \tilde{s}(t) G(s(t) - s(t')) \dot{s}(t')^2 - \frac{\lambda^2 g}{2} \tilde{s}(t) C(s(t) - s(t')) \tilde{s}(t')] \right\}, \quad (3.3)$$

where

$$G(s) = \theta(s) \exp(-s), \quad C(s) = \frac{1}{2} \exp(-|s|). \quad (3.4)$$

Here and in the following, we always disregard contributions of initial disturbances since we are interested in the steady state behavior.

The term with the dimensionless coupling constant  $\alpha$  that arises from the back-reaction of (2.1a) warrants further discussion. LeDW neglect this unpleasant term after arguing that it is irrelevant in the sense of the RG, a reasoning that has to be taken with a grain of salt for this particular term. We will show that this term is redundant instead of irrelevant in the sense of the RG: it does not require an independent renormalization and can therefore be neglected. To this end, we group terms with time-derivatives of  $s$  together and write (suppressing the  $\mathbf{x}$ -dependence for notational convenience)

$$\begin{aligned} \dot{s}(t) + \alpha \int_0^t dt' G(s(t) - s(t')) \dot{s}(t')^2 \\ = \int_0^t dt' \dot{s}(t') [\delta(t - t') + \alpha G(s(t) - s(t')) \dot{s}(t')] \\ = \int_0^{s(t)} ds' [\delta(s(t) - s') + \alpha G(s(t) - s')] \dot{s}'(s') \\ =: \int_0^{s(t)} ds' B(s(t) - s') \dot{s}'(s'), \end{aligned} \quad (3.5)$$

where  $\dot{s}'(\mathbf{x}, s') = \partial_t s(\mathbf{x}, t(s', \mathbf{x}))$  as a shorthand notation.  $t'$  and  $s'$  are connected by the definition (3.1b). Next, we transform the response field  $\tilde{s}$  by letting

$$\tilde{s}(t) \rightarrow \int_t^\infty dt' \tilde{s}(t') K(s(t') - s(t)) \dot{s}(t), \quad (3.6)$$

where

$$K(s) = \delta(s) - \alpha G((1 + \alpha)s) =: (1 + \alpha) D((1 + \alpha)s). \quad (3.7)$$

is the inverse kernel of  $B$ . Then, after the additional rescaling  $(1 + \alpha)s \rightarrow s$ ,  $\tilde{s} \rightarrow \tilde{s} \rightarrow (1 + \alpha)\tilde{s}$ , the reduced response functional takes on the form

$$\mathcal{J}_{red} = \int d^d x \left\{ \int_0^\infty dt \lambda \tilde{s}(t) [\lambda^{-1} \dot{s}(t) - k] - \int_0^{s(t)} ds(t') D(s(t) - s(t')) \nabla^2 s(t') \right. \\ \left. - \frac{\lambda^2 f}{2} \int_0^\infty dt dt' \tilde{s}(t) C(s(t) - s(t')) \tilde{s}(t') \right\}. \quad (3.8)$$

A more detailed account of the steps leading from Eq. (3.3) to (3.8) can be found in Appendix C.

Note that this is the dynamic response functional of the qEW model with an additional retardation of the elastic term  $\sim \nabla^2 s$  described by the function

$$D(s) = \delta(s) - (1 - g/f)G(s). \quad (3.9)$$

The corresponding Langevin equation reads as follows

$$\begin{aligned} \lambda^{-1} \dot{s}(\mathbf{x}, t) &= \int_0^{s(\mathbf{x}, t)} ds(\mathbf{x}, t') D(s(\mathbf{x}, t) - s(\mathbf{x}, t')) \nabla^2 s(\mathbf{x}, t') \\ &\quad + k + \mathcal{F}(\mathbf{x}, t), \\ \langle \mathcal{F}(\mathbf{x}, t) \mathcal{F}(\mathbf{x}', t') \rangle &= f C(s(\mathbf{x}, t) - s(\mathbf{x}, t')) \delta(\mathbf{x} - \mathbf{x}'). \end{aligned} \quad (3.10)$$

Next, we will show that the deviation of the ratio  $g/f$  from 1 does not lead to new renormalizations, and is therefore a redundant, inessential parameter. We consider the cumulant-generation functional

$$\mathcal{W}[\tilde{h}, h] = \ln \left\{ \int \mathcal{D}(\tilde{s}, s) \exp(-\mathcal{J}_{red}[\tilde{s}, s] + (\tilde{h}, \tilde{s}) + (h, s)) \right\}, \quad (3.11)$$

where  $(h, s)$  denotes the integral of the functions  $h$  and  $s$  over space-time, and use the so-called statistical tilt symmetry invariance  $s(\mathbf{x}, t) \rightarrow s(\mathbf{x}, t) + v(\mathbf{x})$ . Exploiting this invariance, it is easy to see that the cumulant-generation functional  $\mathcal{W}$  has the property

$$\mathcal{W}[\tilde{h} + \lambda \bar{D} \nabla^2 v, h] + (h, v) = \mathcal{W}[\tilde{h}, h], \quad (3.12)$$

where  $\bar{D} = \int_0^\infty ds D(s) = g/f$ . A functional derivative with respect to  $v(\mathbf{x})$  produces

$$\int dt \left( \lambda \bar{D} \nabla^2 \frac{\delta \mathcal{W}[\tilde{h}, h]}{\delta \tilde{h}(\mathbf{x}, t)} + h(\mathbf{x}, t) \right) = 0. \quad (3.13)$$

Taking a further functional derivative with respect to  $h(\mathbf{x}', t')$ , we obtain

$$\bar{D} \nabla^2 \int dt \lambda \frac{\delta \mathcal{W}[\tilde{h}, h]}{\delta h(\mathbf{x}', t') \delta \tilde{h}(\mathbf{x}, t)} + \delta(\mathbf{x} - \mathbf{x}') = 0. \quad (3.14)$$

For the Fourier transform of the full response function

$$\begin{aligned} R(\mathbf{x}, t) &= \int_{\mathbf{q}, \omega} R_{\mathbf{q}, \omega} e^{i(\omega t - \mathbf{q} \cdot \mathbf{x})} = \lambda \langle s(\mathbf{x}, t) \tilde{s}(0, 0) \rangle \\ &= \lambda \left. \frac{\delta W[\tilde{h}, h]}{\delta h(\mathbf{x}, t) \delta \tilde{h}(\mathbf{0}, 0)} \right|_{\tilde{h}=h=0}, \end{aligned} \quad (3.15)$$

this leads to

$$R_{\mathbf{q}, \omega=0} = \frac{1}{\bar{D} q^2}. \quad (3.16)$$

Hence, the full static response function does not acquire any additional contributions in perturbation theory, and consequently  $\bar{D}$  does not need any renormalization. Therefore the function  $D(s)$  can be safely approximated by a delta-function  $\delta(s)$ , a step which merely leads

to a resetting of the time-scale by  $\lambda \rightarrow \lambda f/g$  as correctly observed by LeDW. In other words, the effects of the retardation term can be transformed away, and therefore the latter does not contribute to the critical behavior. According to Wegner's canonical classification scheme of field theoretical operators [20] (see also, e.g., Ref. [21]), such a term is called redundant.

By letting  $D(s) \rightarrow \delta(s)$ , we untiately obtain from the reduced dynamic response functional (3.8) the well-known depinning- or qEW-functional

$$\begin{aligned} \mathcal{J}_{qEW} &= \int d^d x \left\{ \int_0^\infty dt \lambda \tilde{s}(t) [\lambda^{-1} \dot{s}(t) - \nabla^2 s(t) - k] \right. \\ &\quad \left. - \frac{\lambda^2}{2} \int_0^\infty dt dt' \tilde{s}(t) \Delta(s(t) - s(t')) \tilde{s}(t') \right\}, \end{aligned} \quad (3.17)$$

with the starting noise correlation

$$\Delta(s) = \frac{f^3}{2g^2} \exp(-|s|), \quad (3.18)$$

Note that this noise correlation shows the characteristic kink for  $s \rightarrow 0$ .

Finally, we would like to comment on the remaining terms that arise from the retardation of the elastic term after shrinking it to a  $\delta$ -function. To study the behavior of the leading such term under the RG, one should determine the renormalization of insertions of a time-bilocal operator of the form

$$\mathcal{O}(\mathbf{x}, t, t') = \tilde{s}(\mathbf{x}, t) O(u(\mathbf{x}, t, t')) \frac{\partial u(\mathbf{x}, t, t')}{\partial t'} \nabla^2 u(\mathbf{x}, t, t') \quad (3.19)$$

where  $u(\mathbf{x}, t, t') = s(\mathbf{x}, t) - s(\mathbf{x}, t')$  with  $O(u)$  denoting some function of  $u$ . Note that if the qEW-model in its original form is renormalizable – as it is generally assumed – such operator-insertions have to be irrelevant.

#### IV. CONCLUDING REMARKS

In summary, we have taken a fresh look at the derivation of field theoretic functional describing C-DP and the recent mapping of C-DP onto qEW.

Our derivation of the C-DP dynamic response functional utilizes the Grassberger-transformation. When it can be applied, the Grassberger-transformation has tangible advantages over the Doi-shift. The field variables produced by the Grassberger-transformation correspond to actual particle densities and their conjugate response field, and we think that they are more natural and more intuitive than the field variables induced by the Doi-shift. The Grassberger-transformation is particularly useful for systems where typical particle configurations correspond to agglomerations of fractal clusters, as is the case for C-DP. Then the scaling dimensions of the density and response fields are typically positive, and terms of higher than harmonic order in the response field are usually irrelevant so that it is justified to truncate

at second order the expansion of the exponentials arising through the transformation. A further advantage is that, if particle-conservation holds as for C-DP, the corresponding symmetry of the response functional is realized linearly. Part of our motivation for the present paper is to highlight the usefulness of the Grassberger-transformation as the concise method to capture the emergent universal description of reaction-diffusion systems in form of a coarse-grained effective stochastic equation of motion, and we hope that its usefulness will be appreciated more in the future. For the reader who is interested in a deeper discussion of the applicability of the Grassberger-transformation to coherent state path integrals in general, we have compiled some additional thoughts in Appendix A.

The profound work by LeDW essentially settled a long-standing issue in statistical physics by mapping C-DP onto qEW. The full response functional for C-DP originally contains an additional term, which is superficially relevant below the upper critical dimension  $d_c = 4$ , and if this term had to be retained, the direct mapping would fail. We show rigorously that this term, which generates a retardation of the elastic interaction in the qEW, does not produce any additional contributions to the static response function at any order in perturbation theory. Thus, the retardation term does not flow under the action of the RG. Consequently, it does not impact the asymptotic scaling behavior of C-DP and therefore can be dropped from the response functional. Note that the retardation term, since it does not flow, it does not flow to zero, in particular, as an irrelevant term would. There are perhaps different interpretations of the notion of irrelevance in the context of the RG. In our personal opinion, the cleanest interpretation or definition is the one given in Wegner's classification scheme, and according to this classification, the retardation term is redundant. Irrelevant and redundant terms, respectively, behave qualitatively differently under the action of the RG, however, ultimately both can and should be discarded from minimal field theoretic models. Fortunately, after the dust settles, the omission of the retardation term is justified, albeit on different grounds, and the outcome of the LeDW mapping stands correct.

Finally, we would like to point out that the appearance of the retardation term in the qEW functional is not merely a consequence of the mapping from C-DP but that this term is inherent in the qEW model itself. When one studies fluctuation effects based on the original qEW functional that has no retardation term (except for noise contributions), diagrammatic perturbation theory does eventually produce such a term, and this term is exactly of the form discussed above. The qEW model is generally accepted as being renormalizable as it stands. One can show that the retardation term, though marginal on dimensional grounds, does not require an independent renormalization as a consequence of statistical tilt-symmetry, and hence it is redundant and can be omitted. In other words, the redundant retardation term

is native to qEW.

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## Appendix A: Doi-shift versus Grassberger-transformation

It is well known that particle-reactions and -diffusion modeled by master-equations, and described by “second-quantized” Fock-space methods, can be conveniently formulated as coherent-state path-integrals (CSPI) [22–27]. For simplicity, we consider here only one sort of particles,  $A$ , with reactions

$$kA \xrightarrow{r_{k,l}} lA \quad (\text{A1})$$

and reaction-rates  $r_{k,l}$ . After applying a (naive) continuum limit, the corresponding CSPI-action with an additional diffusion-term is given by

$$\mathcal{S} = \int d^d x \int_{-\infty}^{\infty} dt \left\{ (\hat{a} - 1) \partial_t a + \lambda \nabla \hat{a} \cdot \nabla a - \sum_{k,l} r_{k,l} (\hat{a}^l - \hat{a}^k) a^k \right\}. \quad (\text{A2})$$

Here, the fields  $\hat{a}$  and  $a$  correspond to the coherent-state eigenvalues of the bosonic creation and annihilation operators of the Fock-space with initial and final conditions  $\hat{a}(\infty) = 1$ ,  $a(-\infty) = 0$ . It is important to understand that the complex field  $a$  is not the particle-density which is given by  $n = \hat{a}a$ , a real semipositive quantity. Following Doi, usually a field shift according to  $\hat{a} = 1 + \tilde{a}$  is applied. This Doi-shift results in

$$\mathcal{S} = \int d^d x \int_{-\infty}^{\infty} dt \left\{ \tilde{a} \partial_t a + \lambda \nabla \tilde{a} \cdot \nabla a - \tilde{a} \sum_{k,l} (l - k) r_{k,l} a^k - \frac{\hat{a}^2}{2} \sum_{k,l} (l + k - 1)(l - k) r_{k,l} a^k + \dots \right\}, \quad (\text{A3})$$

where the series stops with the quadratic term  $\hat{a}^2$  if  $l, k \leq 2$ . In a corresponding Langevin description, this quadratic term is often interpreted as a noise term. Then one gets the well known result: branching ( $l > k$ ) leads to real noise whereas annihilation ( $l < k$  with exception of  $k = 1$ ) leads to imaginary noise which is to interpreted as a first passage problem [27]. This type of behavior is characteristic for processes where random walkers sparsely distributed in space which meet and react from time to time, however, it is not so for systems of clusters of, in general, fractal particle-agglomerations (typical for percolating processes). For the latter type of systems, it is more appropriate to switch to the density  $n$  as the



fundamental variable. This is achieved by Grassbergers quasi-canonical transformation [28–30]

$$a = n \exp(-\tilde{n}), \quad \hat{a} = \exp(\tilde{n}), \quad (\text{A4})$$

which is similar to an inverse Cole-Hopf transformation. Note that a creation of a state with  $\rho_i$  particles in cell  $i$  of a spatially distributed system corresponds in the CSPI to an insertion of the product

$$\prod_i \hat{a}_i^{\rho_i} = \exp\left(\sum_i \rho_i \tilde{n}_i\right) \rightarrow \exp\left(\int d^d x \rho(\mathbf{x}) \tilde{n}(\mathbf{x})\right), \quad (\text{A5})$$

in the continuum limit. Hence, coarse-graining is simply performed by neglecting higher Fourier components of  $\tilde{n}$ . We will show in a moment that this product can be simply interpreted as a creation process

$$0 \xrightarrow{\rho} A \quad (\text{A6})$$

in the transformed action (A2)  $\mathcal{S}[\hat{a}, a] \rightarrow \mathcal{J}[\tilde{n}, n]$  of the CSPI.

Applying the Grassberger-transformation (A4), we obtain after some partial integrations using  $\tilde{n}(\infty) = n(-\infty) = 0$

$$\begin{aligned} \mathcal{J}[\tilde{n}, n] &= \int d^d x \int_{-\infty}^{\infty} dt \left\{ \tilde{n} \partial_t n + \lambda \nabla \tilde{n} \cdot \nabla n - \lambda n (\nabla \tilde{n})^2 \right. \\ &\quad \left. - \sum_{k,l} r_{k,l} (\exp((l-k)\tilde{n}) - 1) n^k \right\} \\ &= \int d^d x \int_{-\infty}^{\infty} dt \left\{ \tilde{n} \partial_t n + \lambda \nabla \tilde{n} \cdot \nabla n - \lambda n (\nabla \tilde{n})^2 \right. \\ &\quad \left. - \tilde{n} \sum_{k,l} (l-k) r_{k,l} n^k - \frac{\tilde{n}^2}{2} \sum_{k,l} (l-k)^2 r_{k,l} n^k \right. \\ &\quad \left. + \dots \right\}. \end{aligned} \quad (\text{A7})$$

Note that the particle insertion (A5) into the path-integral corresponds to a particle-creation process (A6) that leads to a term with  $k = 0$  and  $l = 1$  in the functional (A7). The expansion of the exponential in  $\mathcal{J}$  is analogous to a Kramers-Moyal expansion of the master-equation. Skipping all the terms higher than the quadratic term  $\sim \tilde{n}^2$  leads to the well-known dynamic-response functional [31–33] of reaction-diffusion systems represented by a Langevin equation (in Ito-interpretation)

$$\partial_t n = \lambda \nabla^2 n + R(n) + \zeta, \quad (\text{A8})$$

where the rate is  $R(n) = \sum_{k,l} (l-k) r_{k,l} n^k =: \sum_k r_k n^k$ , and  $\zeta$  is a real Gaussian noise with correlator

$$\langle \zeta(\mathbf{x}, t) \zeta(\mathbf{x}', t') \rangle = [Q(n) - \lambda \nabla n \nabla] \delta(\mathbf{x} - \mathbf{x}') \delta(t - t') \quad (\text{A9})$$

with  $Q(n) = \sum_{k,l} (l-k)^2 r_{k,l} n^k =: \sum_k q_k n^k$ .

The truncation of the expansion at second order deserves some further scrutiny [26]. Let us carefully examine the behavior of the expansion under coarse graining.

To this end, we assess the scaling behavior of the fields near the upper dimension above which a simple mean-field approximation is correct. Naively, the response field  $\tilde{n}$  is dimensionless, and the particle-density scales as  $n \sim \mu^d$  where  $\mu$  is an inverse length scale. Thus, the reaction-constants scale naively as  $r_k \sim q_k \sim \lambda \mu^{2-(k-1)d}$ . Let  $k_0$  be the lowest  $k$  so that  $r_k \neq 0$ , and  $k_1$  the lowest  $k$  with  $q_k \neq 0$ . Then  $k_0 \geq k_1$ , and a rescaling of the fields with  $\mu$  so that  $\tilde{n} n^{k_0} \sim \tilde{n}^2 n^{k_1}$  and  $\tilde{n} n \sim \mu^d$  leads to

$$\tilde{n} \sim \mu^{d(k_0-k_1)/(k_0-k_1+1)}, \quad n \sim \mu^{d/(k_0-k_1+1)}. \quad (\text{A10})$$

Thus, if  $k_0 > k_1$  ( $k_0 = k_1 + 1$  in typical cases) both fields receive positive scaling dimensions ( $\tilde{n} \sim n \sim \mu^{d/2}$  if  $k_0 = k_1 + 1$ ), and all higher order monomials in  $R(n)$  and  $Q(n)$  as well as all contributions proportional to  $\tilde{n}^l$  with exponents  $l > 2$  are irrelevant and contribute only corrections to the leading scaling behavior generated by the relevant leading terms  $\tilde{n} n^{k_0}$  and  $\tilde{n}^2 n^{k_1}$ . Even the diffusional noise is sub-leading if  $k_1 < 2$ . This reasoning justifies the truncation of the expansion at second order. It shows that the application of the Grassberger-transformation in such cases is the concise method to capture the emergent universal description of reaction-diffusion systems.

However, if  $k_0 = k_1$ , typical for annihilation reactions, the Grassberger-transformation to density fields is not applicable, and one has to deal with the original formulation of the CSPI. The superficial scaling behavior is correct, of course, above and at a critical dimension  $d_c$ , and below  $d_c$ , perturbational corrections appear. The critical dimension is determined by the similarity  $\tilde{n} n^{k_0} \sim \tilde{n}^2 n^{k_1} \sim \mu^{d_c+2}$ . One obtains

$$d_c = 2 \frac{(k_0 - k_1 + 1)}{(k_0 - 1)}. \quad (\text{A11})$$

## Appendix B: Doi-shift, truncations, imaginary and real diffusional noise

Here, we present an alternate derivation of the dynamical response functional, Eq. (2.7), that uses the Doi-shift instead of the Grassberger-transformation. Our motivation here is twofold. First, we think that it is instructive to see the two approaches side by side so that one can compare them in a specific example. Most readers will agree with us that the route via the Doi-shift is significantly less intuitive and more cumbersome than the one taken in the main text. Second, we feel that the way the derivation of the original, full (with diffusion of the A-particles) C-DP functional has been presented in the literature might leave the reader wondering where some of the key terms come from. Thus, we review here some of the essential steps involved in the derivation.

Our starting point is the CSPI-action, Eq. (2.2), augmented by terms that arise when the diffusion of A-

particles is permitted [1]:

$$\begin{aligned} \mathcal{S} &= \int d^d x dt \left\{ (\hat{a} - 1) \partial_t a + (\hat{b} - 1) \partial_t b + D \nabla \hat{b} \cdot \nabla b + D' \nabla \hat{a} \cdot \nabla a \right. \\ &\quad \left. - k_1 (\hat{b}^2 - \hat{a} \hat{b}) ab - k_2 (\hat{a} \hat{b} - \hat{b}^2) b^2 - k_3 (\hat{a} - \hat{b}) b \right\} \\ &= \int d^d x dt \left\{ \tilde{a} \partial_t a + \tilde{b} \partial_t b + D \nabla \tilde{b} \cdot \nabla b + D' \nabla \tilde{a} \cdot \nabla a \right. \\ &\quad \left. + (\tilde{b} - \tilde{a}) [k_3 + k_2 b - k_1 a] b - (\tilde{b} - \tilde{a}) \tilde{b} [k_1 a - k_2 b] b \right\}. \end{aligned} \quad (\text{B1})$$

Now, let's focus on the noise term (the one of second order in the fields with the tilde). In this term, we truncated as follows,

$$k_1 a - k_2 b = k_1 c_0 + \dots, \quad (\text{B2})$$

where  $c_0$  is the constant initial value of  $c$ . Setting  $\tilde{b}' = \tilde{b} - \tilde{a}$ , the truncated noise term becomes

$$\begin{aligned} & - (\tilde{b}'^2 + \tilde{a} \tilde{b}') b k_1 c_0 = - (\tilde{b}'^2 + \tilde{a} \tilde{b}') \\ & = - (\tilde{b}', \tilde{a}) \begin{pmatrix} 1 & 1/2 \\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} \tilde{b}' \\ \tilde{a} \end{pmatrix} b k_1 c_0. \end{aligned} \quad (\text{B3})$$

Note that the matrix on the right hand side has a positive and a negative eigenvalue implying that the noise described by this noise term has both real and imaginary components as recently pointed out by Wiese [27]. Note also that the problematic term  $\tilde{a} \tilde{b}'$  that originally appeared in Ref. [1] was absent in the follow-up paper [2], where a real diffusional noise term appeared instead. This brings up the question of how imaginary noise can become real diffusional noise.

To answer this question, let us substitute

$$a' = a + c_0 \tilde{a} \quad (\text{B4})$$

and integrate out  $\tilde{a} \partial_t \tilde{a} = 1/2 \partial_t \tilde{a}^2$ . Then the CSPI action becomes

$$\begin{aligned} \mathcal{S} &= \int d^d x dt \left\{ \tilde{a} \partial_t (a' + b) + \tilde{b}' \partial_t b + D \nabla \tilde{b}' \cdot \nabla b \right. \\ &\quad \left. + \nabla \tilde{a} \cdot (D \nabla b + D' \nabla a') + \tilde{b}' [k_3 + k_2 b - k_1 a'] b \right. \\ &\quad \left. - k_1 c_0 \tilde{b}'^2 b - D' c_0 (\nabla \tilde{a})^2 \right\}, \end{aligned} \quad (\text{B5})$$

where the real diffusional noise has replaced the imaginary noise.

To proceed from here to our dynamical response functional  $\mathcal{J}$ , we set  $a' + b = c$ ,  $\tilde{a} = \tilde{c}$ ,  $b = n$ ,  $\tilde{b}' = n$  and  $D' = 0$ . Up to a trivial redefinition of coefficients and the source terms that we still need to add, this produces  $\mathcal{J}$  as given in Eq. (2.7).

Finally, let us compile what the switch from  $a$  to  $a'$  means on the level of the particle densities  $n_A$  and  $n_B$  and the total particle density  $c$ . Expansion, truncation

and this switch give

$$n_A = e^{\tilde{n}_A} a = a + \tilde{a} a + \dots \quad (\text{B6a})$$

$$= a + \tilde{a} (c_0 + \dots) + \dots = a' + \dots, \quad (\text{B6b})$$

$$n_B = e^{\tilde{n}_B} b = b + \dots, \quad (\text{B6c})$$

$$c = n_A + n_B = a' + b + \dots, \quad (\text{B6d})$$

in full agreement with the approach based on the Grassberger-transformation.

As pointed out earlier, we feel that the approach based on the Grassberger-transformation is much clearer and more elegant than the one based on the Doi-shift. After all, the essential physical ingredient of the model is that total particle density fluctuates about its fixed finite initial value  $c_0$ , and the resulting theory is based on an expansion in the fluctuations of the density fields  $c$ ,  $n$  and their corresponding response fields. The natural advantage of the Grassberger-transformation hereby is that it describes physical densities from the onset.

### Appendix C: Derivation of the reduced response functional $\mathcal{J}_{red}$

In this Appendix, we provide some details on how to proceed from the reduced response functional  $\mathcal{J}_{red}$  as stated in Eq. (3.3) to its form stated in Eq. (3.8). We focus on the most difficult term, i.e., the last term in these two equations. We assume the usual rules of Ito-calculus, in particular

$$\theta(s) = \theta(s+0), \quad \delta(s) = \delta(s+0). \quad (\text{C1})$$

Applying Laplace-transformation to the propagator given in Eq. (3.4), we have

$$\hat{G}(z) = \int_0^\infty ds e^{-zs} G(s) = \frac{1}{z+1}. \quad (\text{C2})$$

For the Laplace-transformation of the kernel  $B(s)$  appearing in the equation of motion, Eq. (3.5), this implies

$$\hat{B}(z) = 1 + \frac{\alpha}{z+1} = \frac{z+1+\alpha}{z+1}. \quad (\text{C3})$$

Because Laplace-transformation reduces convolution integrals to simple multiplications, it produces

$$\hat{K}(z) = \frac{z+1}{z+1+\alpha} = 1 - \frac{\alpha}{z+1+\alpha} \quad (\text{C4})$$

for the inverse of the kernel  $B(s)$ . Note that this is nothing but the Laplace-transformation of Eq. (3.7).

Now, we are in position to apply Laplace-transformation to the kernel  $C(s)$  in Eq. (3.3). Writing the correlator as

$$C(s-s') = \int ds_0 G(s-s_0) G(s'-s_0), \quad (\text{C5})$$

we have

$$\begin{aligned}
& g \int \int dt dt' \tilde{s}(t) \tilde{s}(t') C(s(t) - s(t')) \\
& = g \int ds_0 \left( \int dt \tilde{s}(t) G(s(t) - s_0) \right) \\
& \times \left( \int dt' \tilde{s}(t') G(s(t') - s_0) \right). \quad (C6)
\end{aligned}$$

Under the transformation (3.6), the individual factors on the right hand side of Eq. (C6) transform as

$$\begin{aligned}
& \int dt \tilde{s}(t) G(s(t) - s_0) \\
& \rightarrow \int dt' \tilde{s}(t') \int dt K(s(t') - s(t)) \dot{s}(t) G(s(t) - s_0) \\
& = \int dt' \tilde{s}(t') \int ds K(s(t') - s) G(s - s_0). \quad (C7)
\end{aligned}$$

Upon Laplace-transformation, the convolution of  $K$  und  $G$  turns into a product of  $\hat{K}$  und  $\hat{G}$ :

$$\hat{K}(z) \hat{G}(z) = \frac{1}{z + 1 + \alpha}. \quad (C8)$$

This is the same as the result of Laplace-transformation applied to  $G((1 + \alpha)s)$ . Thus, we have

$$\begin{aligned}
& g \int \int dt dt' \tilde{s}(t) \tilde{s}(t') C(s(t) - s(t')) \\
& \rightarrow \frac{g}{1 + \alpha} \int \int dt dt' \tilde{s}(t) \tilde{s}(t') C((1 + \alpha)(s(t) - s(t'))) . \quad (C9)
\end{aligned}$$

Finally, applying the rescaling  $(1 + \alpha)s \rightarrow s$ ,  $\tilde{s} \rightarrow (1 + \alpha)\tilde{s}$  and setting  $(1 + \alpha) = f/g$  completes the journey from Eq. (3.3) to Eq. (3.8).

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